

# THE BAR AUTOMORPHISM IN QUANTUM GROUPS AND GEOMETRY OF QUIVER REPRESENTATIONS

PHILIPPE CALDERO AND MARKUS REINEKE

ABSTRACT. Two geometric interpretations of the bar automorphism in the positive part of a quantized enveloping algebra are given. The first is in terms of numbers of rational points over finite fields of quiver analogues of orbital varieties; the second is in terms of a duality of constructible functions provided by preprojective varieties of quivers.

## 1. INTRODUCTION

The canonical basis  $\mathcal{B}$  of the positive part  $U_v(\mathfrak{g})^+$  of the quantized enveloping algebra of a semisimple Lie algebra  $\mathfrak{g}$ , constructed by G. Lusztig [4], has many favourable properties, like for example inducing bases in all the finite dimensional irreducible representations of  $\mathfrak{g}$  simultaneously.

The basis  $\mathcal{B}$  can be characterized algebraically by its elements being fixed under the so-called bar automorphism of  $U_v(\mathfrak{g})^+$ , and by admitting a unitriangular base change to the PBW-type bases.

The Hall algebra approach to quantum groups [11] provides a realization of certain specializations  $U_q(\mathfrak{g})^+$  via a convolution product for constructible functions on varieties  $R_d$  parametrizing representations of Dynkin quivers. In this realization, the elements of PBW-type bases correspond to characteristic functions of orbits in  $R_d$  under a natural action of an algebraic group  $G_d$ , whereas the elements of  $\mathcal{B}$  correspond to constructible functions naturally associated to the intersection cohomology complexes of the closures of  $G_d$ -orbits.

It is therefore natural to also ask for interpretations of the bar automorphism in terms of the geometry of the varieties  $R_d$ , since this automorphism plays a central role in defining the canonical basis algebraically.

In the present paper, two such interpretations are given. In the geometric setup of [6], analogues of orbital varieties in the quiver context are constructed. These parametrize quiver representations fixing certain flags, and their numbers of rational points over finite fields are shown (Theorem 3.4) to give essentially the coefficients  $\Omega_{M,N}$  of the bar automorphism on a PBW-type basis of  $U_q(\mathfrak{g})^+$ . The key ingredient in deriving this result in section 3 is a generalization (Corollary 2.6) of a very useful formula [8] by C. Riedmann, relating numbers of filtrations of quiver representations over finite fields to cardinalities of orbital varieties; this generalization, together with the construction of the orbital varieties, is given in section 2.

The second interpretation starts from a duality between constructible functions on the varieties  $R_d$  provided by the preprojective varieties of a quiver, also used

---

*Date:* February 1, 2008.

by G. Lusztig [5]. The coefficients  $\Omega_{M,N}$  are shown (Proposition 4.2) to be essentially given by a convolution operator derived from a certain twisted version of this duality, constructed in section 4.

**Acknowledgments:** The present work was done during a stay of both authors at the University of Antwerp in the framework of the Priority Programme "Non-Commutative Geometry" of the European Science Foundation.

## 2. A GENERALIZATION OF RIEDTMANN'S FORMULA

2.1. Let  $k$  be a field. We fix a finite quiver  $Q$  with set of vertices  $I$  and set of arrows  $Q_1$ , whose underlying unoriented graph is a disjoint union of Dynkin diagrams of type  $A_n$ ,  $D_n$ ,  $E_6$ ,  $E_7$  or  $E_8$ . Fixing a dimension type  $d = \sum_i d_i i \in \mathbb{N}I$ , define an  $I$ -graded vector space  $V_d := \bigoplus_{i \in I} k^{d_i}$ . For any subquotient  $W$  of  $V_d$  compatible with the grading, we set  $\underline{\dim}(W) := \sum_i (\dim W_i) i \in \mathbb{N}I$ , where  $W_i$  denotes the  $i$ -component of  $W$ .

Set  $R_d := R_d(Q) := \bigoplus_{\alpha: i \rightarrow j} \text{Hom}(k^{d_i}, k^{d_j})$  and  $G_d := \prod_i GL_{d_i}(k)$ . The algebraic group  $G_d$  acts linearly on the affine space  $R_d$  by  $(g \cdot M)_{\alpha: i \rightarrow j} := g_j M_{\alpha} g_i^{-1}$ .

2.2. Let  $\nu$  be a positive integer, and let  $\mathbf{d} = (d^1, \dots, d^\nu)$  be a  $\nu$ -tuple of elements in  $\mathbb{N}I$  such that  $d = \sum_s d^s$ . Let  $\mathcal{F}_{\mathbf{d}}$  be the set of filtrations  $F^* = (0 = F^\nu \subset \dots \subset F^1 \subset F^0 = V_d)$  of the graded space  $V_d$  such that  $\underline{\dim} F^{s-1}/F^s = d^s$ ,  $1 \leq s \leq \nu$ . The action of the group  $G_d$  on  $V_d$  provides a transitive action of  $G_d$  on  $\mathcal{F}_{\mathbf{d}}$ .

We fix an arbitrary filtration  $F_0^*$  in  $\mathcal{F}_{\mathbf{d}}$ . Choosing successive complements, we can assume that  $V_d$  has a direct sum decomposition  $V_d = \bigoplus_{s=1}^\nu V_d^s$  (as  $I$ -graded  $k$ -space), such that  $F_0^s = \bigoplus_{t > s} V_d^t$  for  $s = 1, \dots, \nu$ . This induces a decomposition  $R_d = \bigoplus_{s,t=1}^\nu R_d^{s,t}$  by setting  $R_d^{s,t} = \bigoplus_{\alpha: i \rightarrow j} \text{Hom}((V_d^s)_i, (V_d^t)_j)$ . Let  $P_{\mathbf{d}}$  be the stabilizer of  $F_0^*$  in  $G_d$ . This is a parabolic subgroup of  $G_d$ , providing an identification  $\mathcal{F}_{\mathbf{d}} \simeq G_d/P_{\mathbf{d}}$ .

We say that an element  $M$  in  $R_d$  is compatible with a filtration  $F^*$  in  $\mathcal{F}_{\mathbf{d}}$  if  $M_\alpha(F_i^s) \subset F_j^s$  for any arrow  $\alpha: i \rightarrow j$  in  $Q_1$  and any  $s = 1, \dots, \nu$ . Set

$$X_{\mathbf{d}} := \{(M, F^*), M \in R_d, F^* \in \mathcal{F}_{\mathbf{d}}, M \text{ compatible with } F^*\} \subset R_d \times \mathcal{F}_{\mathbf{d}}.$$

The diagonal action of the group  $G_d$  on  $R_d \times \mathcal{F}_{\mathbf{d}}$  respects  $X_{\mathbf{d}}$ , and the projections  $p_1$  and  $p_2$  on  $R_d$  and  $\mathcal{F}_{\mathbf{d}}$ , respectively, are  $G_d$ -equivariant.

Set  $Y_{\mathbf{d}} := p_2^{-1}(F_0^*)$ , which will be identified with its image  $p_1(Y_{\mathbf{d}})$  in  $R_d$ . We have the identification

$$(2.1) \quad G_d \times^{P_{\mathbf{d}}} Y_{\mathbf{d}} \simeq X_{\mathbf{d}}, \quad \overline{(g, y)} := \{(gp, p^{-1} \cdot y), p \in P_{\mathbf{d}}\} \mapsto (g \cdot y, gF_0^*).$$

Denote by  $U_{\mathbf{d}}$  the unipotent radical of  $P_{\mathbf{d}}$  and set  $\tilde{X}_{\mathbf{d}} := G_d \times^{U_{\mathbf{d}}} Y_{\mathbf{d}}$ . Let  $\pi$  be the natural projection  $\tilde{X}_{\mathbf{d}} \rightarrow G_d \times^{P_{\mathbf{d}}} Y_{\mathbf{d}} \simeq X_{\mathbf{d}}$ . The Levi decomposition gives  $P_{\mathbf{d}} = L_{\mathbf{d}} U_{\mathbf{d}}$ , where  $L_{\mathbf{d}}$  is the Levi of the parabolic  $P_{\mathbf{d}}$ . We have a diagonal action of  $L_{\mathbf{d}}$  on  $\tilde{X}_{\mathbf{d}}$  defined by  $l.(g, y) := (gl^{-1}, l.y)$ , and a left action of  $G_d$  on  $\tilde{X}_{\mathbf{d}}$ . These two actions commute.

The above direct sum decomposition  $V_d = \bigoplus_s V_d^s$  provides a surjection  $\zeta: R_d \rightarrow \prod_s R_{d^s}$ , which restricts to a surjection  $\zeta: Y_{\mathbf{d}} \rightarrow \prod_s R_{d^s}$ . This clearly defines a surjection  $\tilde{\zeta}: G_d \times^{U_{\mathbf{d}}} Y_{\mathbf{d}} \rightarrow \prod_s R_{d^s}$  by projecting on the second factor. We obtain

the following diagram.

$$(2.2) \quad \begin{array}{ccc} \prod_s R_{ds} & \xleftarrow{\tilde{\zeta}} & \tilde{X}_{\mathbf{d}} \\ & & \downarrow \pi \\ R_d & \xleftarrow{p_1} & X_{\mathbf{d}} \xrightarrow{p_2} \mathcal{F}_{\mathbf{d}} \simeq G_d/P_{\mathbf{d}} \end{array}$$

The following lemma follows immediately from the definitions:

**Lemma 2.1.** *The morphism  $\tilde{\zeta}$  commutes with the action of  $L_{\mathbf{d}}$ . The morphisms  $\pi$ ,  $p_1$  and  $p_2$  commute with the action of  $G_d$ .*

2.3. Let  $\text{mod } kQ$  be the category of finite dimensional  $k$ -representations of  $Q$ . For a representation  $X$  in  $\text{mod } kQ$  of dimension type  $\underline{\dim}(X) = d$ , we denote by  $\mathcal{O}_X$  the corresponding  $G_d$ -orbit in  $R_d$ . The  $G_d$ -orbits are in bijection with the isoclasses of representations of dimension type  $d$  in  $\text{mod } kQ$  by definition. We denote by  $\text{mod } kQ$  this set of isoclasses, and  $\overline{X}$  will be the isoclass corresponding to the representation  $X$ .

We now suppose that  $\nu$  is the number of isoclasses of indecomposable representations in  $\text{mod } kQ$  (which coincides with the number of positive roots of the root system corresponding to  $Q$  by Gabriel's Theorem), and we denote by  $I_s$  for  $1 \leq s \leq \nu$  these indecomposable representations, ordered such that

$$(2.3) \quad \text{Hom}_Q(I_t, I_s) = 0 \text{ for } 1 \leq s < t \leq \nu,$$

where  $\text{Hom}_Q$  denotes the space of homomorphisms in the category  $\text{mod } kQ$  (such an ordering exists since the category  $\text{mod } kQ$  is directed; for this and other facts on  $\text{mod } kQ$  see, for example, [9]).

We fix two representations  $N$  and  $M$  in  $R_d$ . In the following, we define analogues of the varieties introduced in the previous section, naturally associated to  $N$  and  $M$ .

Let  $N = \bigoplus_s N_s$  be the unique decomposition of the representation  $N$  such that  $N_s$  is isomorphic to a direct sum of copies of  $I_s$  for  $1 \leq s \leq \nu$ . We can suppose without loss of generality that the spaces  $N_s$  are compatible with the direct sum decomposition  $V_d = \bigoplus_s V_d^s$ , so that  $N_s$  belongs to  $R_d^{s,s}$ .

We define  $X_{\overline{N}}$  as the set of pairs  $(P, F^*)$  in  $X_{\mathbf{d}}$  such that the representation induced by  $P$  on  $F^{s-1}/F^s$  is isomorphic to  $N_s$  for any  $s = 1, \dots, \nu$ . Let  $X_{\overline{N}}^{\overline{M}}$  be the subset of pairs  $(P, F^*)$  in  $X_{\overline{N}}$  such that  $P$  belongs to  $\mathcal{O}_M$ . Set  $Y_{\overline{N}} := \zeta^{-1}(\prod_s \mathcal{O}_{N_s}) \subset Y_{\mathbf{d}}$ . Again, the following lemma follows immediately from the definitions:

**Lemma 2.2.** *Via the identification 2.1, we have*

$$X_{\overline{N}} \simeq G_d \times^{P_{\mathbf{d}}} Y_{\overline{N}} \text{ and } X_{\overline{N}}^{\overline{M}} \simeq G_d \times^{P_{\mathbf{d}}} (Y_{\overline{N}} \cap \mathcal{O}_M).$$

We define

$$\tilde{X}_{\overline{N}}^{\overline{M}} := G_d \times^{U_{\mathbf{d}}} (Y_{\overline{N}} \cap \mathcal{O}_M) \subset \tilde{X}_{\overline{N}} := G_d \times^{U_{\mathbf{d}}} Y_{\overline{N}} \subset X_{\mathbf{d}}.$$

The left action of  $G_d$  and the diagonal action of  $L_{\mathbf{d}}$  both stabilize these varieties.

2.4. From now on, we suppose that  $k$  is a finite field with  $q$  elements. For any representation  $P$  in  $R_d$ , we denote by  $\text{Aut}(P) \subset G_d$  the stabilizer of  $P$  (which coincides with the automorphisms of  $P$  as an object in  $\text{mod } kQ$  by definition). With the notation of the previous section, let  $Y_N$  be the fiber of  $\zeta$  over the point  $(N_s)$ , that is,  $Y_N := \zeta^{-1}((N_s)) \subset Y_{\mathbf{d}}$ . Note that  $U_{\mathbf{d}}$  acts on  $Y_N$ .

**Proposition 2.3.** *With notation as above, we have*

$$|p_1^{-1}(M) \cap X_{\overline{N}}| = \frac{|\text{Aut}(M)| \cdot |Y_{\overline{N}} \cap \mathcal{O}_M|}{(\prod_s |\text{Aut}(N_s)|) \cdot |U_{\mathbf{d}}|}.$$

*Proof.* First we have

$$\pi^{-1} p_1^{-1}(\mathcal{O}_M) \cap \tilde{X}_{\overline{N}} = \tilde{X}_{\overline{N}}^{\overline{M}} \simeq G_d \times^{U_{\mathbf{d}}} (Y_{\overline{N}} \cap \mathcal{O}_M).$$

Since  $p_1 \pi$  commutes with the  $G_d$ -action, we obtain

$$\begin{aligned} |\pi^{-1} p_1^{-1}(M) \cap \tilde{X}_{\overline{N}}| &= \frac{|G_d| \cdot |Y_{\overline{N}} \cap \mathcal{O}_M|}{|\mathcal{O}_M| \cdot |U_{\mathbf{d}}|} \\ &= \frac{|\text{Aut}(M)| \cdot |Y_{\overline{N}} \cap \mathcal{O}_M|}{|U_{\mathbf{d}}|}. \end{aligned}$$

From Lemma 2.1, we conclude

$$\begin{aligned} |p_1^{-1}(M) \cap X_{\overline{N}}| &= \frac{|\text{Aut}(M)| \cdot |Y_{\overline{N}} \cap \mathcal{O}_M|}{|L_{\mathbf{d}}| \cdot |U_{\mathbf{d}}|} \\ &= \frac{|\text{Aut}(M)| \cdot (\prod_s |\mathcal{O}_{N_s}|) \cdot |Y_{\overline{N}} \cap \mathcal{O}_M|}{|L_{\mathbf{d}}| \cdot |U_{\mathbf{d}}|}. \end{aligned}$$

This implies the proposition.  $\square$

2.5. In this section, we give a more precise version of Proposition 2.3. Let  $\mathfrak{g}_d := \bigoplus_i \mathfrak{gl}(k^{d_i})$  be the Lie algebra of the group  $G_d$ . The components of an element  $\xi$  in  $\mathfrak{g}_d$  will be denoted by  $\xi_i$ . Let  $\mathfrak{u}_{\mathbf{d}} \subset \mathfrak{g}_d$  be the Lie algebra of  $U_{\mathbf{d}}$ . The differential of the morphism  $G_d \rightarrow G_d.N$  gives rise to a morphism of vector spaces  $\phi : \mathfrak{g}_d \rightarrow R_d$  given by  $\phi(\xi)_{\alpha:i \rightarrow j} = \xi_j N_{\alpha} - N_{\alpha} \xi_i$ .

**Lemma 2.4.** *The morphism  $\phi$  has the following properties :*

- (i)  $\text{Ker}(\phi) = \text{End}_Q(N)$ ,
- (ii)  $\text{Im}(\phi) = T_N$ , where  $T_N := T_N(\mathcal{O}_N)$  is the tangent space to  $\mathcal{O}_N$  at the point  $N$ ,
- (iii)  $\text{Im}(\phi)$  is compatible with the decomposition  $R_d = \bigoplus_{s,t} R_d^{s,t}$ , and it contains the subspace  $\bigoplus_{s \leq t} R_d^{s,t}$ ,
- (iv) the restriction of  $\phi$  to  $\mathfrak{u}_{\mathbf{d}}$  is injective.

*Proof.* (i) follows from the definition of  $\phi$  and of the category  $\text{mod } kQ$ . (ii) is clear. The first assertion of (iii) follows from the fact that  $\phi$  decomposes into a direct sum  $\phi = \bigoplus_{s,t} \phi_{s,t}$ , where

$$(2.4) \quad \phi_{s,t} : \bigoplus_i \text{Hom}_k((V_d^s)_i, (V_d^t)_i) \rightarrow R_d^{s,t}.$$

The second assertion is [4, Lemma 10.4]. To prove (iv), we remark that  $\phi|_{\mathfrak{u}_{\mathbf{d}}} = \bigoplus_{s > t} \phi_{s,t}$  which implies that  $\text{Ker} \phi|_{\mathfrak{u}_{\mathbf{d}}} = 0$  by formula 2.3 and (i).  $\square$

We now consider the affine space  $Y_N$  identified with its tangent vector space  $T_N(Y_N)$  by  $x \mapsto x + N$ .

**Proposition 2.5.** *The space  $Y_N$  contains  $\phi(\mathfrak{u}_{\mathbf{d}})$ . Let  $E_N$  be any complement of  $\phi(\mathfrak{u}_{\mathbf{d}})$  in  $Y_N$  which is compatible with the decomposition  $\bigoplus R_d^{s,t}$  of  $R_d$ . Then,*

- (i) *the space  $E_N$  is a direct summand of  $T_N(\mathcal{O}_N)$  in  $R_d$ ,*
- (ii)  *$E_N$  is a transversal slice for the action of  $U_{\mathbf{d}}$  on  $Y_N$ .*

*Proof.* First we remark that, by construction, the space  $Y_N$  is compatible with the decomposition  $R_d = \bigoplus_{s,t} R_d^{s,t}$ . Thus, a complement  $E_N$  as above exists by Lemma 2.4 (iii).

(i) is a consequence of Lemma 2.4 (ii), (iii) and the decomposition 2.4 of  $\phi$ .

Now we prove (ii). Fix a complement  $E_N$  and let  $X$  be in  $E_N$ . We claim that  $X$  is the unique element of  $E_N$  in the orbit  $U_{\mathbf{d}} \cdot X$ . Suppose that  $Y = U \cdot X \in E_N$ , with  $U \in U_{\mathbf{d}}$ . The component of  $U$  (induced by the decomposition  $V_d = \bigoplus_s V_d^s$ ) belonging to  $\bigoplus_i \text{Hom}((V_d^s)_i, (V_d^t)_i)$  is denoted by  $U_{s,t}$  for  $s \geq t$ . Note that  $U_{s,s}$  is the identity for all  $s = 1, \dots, \nu$ .

We prove by induction on  $s - t > 0$  that  $U_{s,t} = 0$ , for  $s > t$ . Fix a pair  $(s, t)$  for  $1 \leq t < s \leq \nu$ . It is easily seen by a weight argument that the induction hypothesis implies that the component  $Y_{s,t}$  of  $Y$  has the following form :  $Y_{s,t} = U_{s,t}N_t - N_sU_{s,t}$ . So, we have  $U_{s,t}N_t - N_sU_{s,t} = Y_{s,t} \in E_N$  by the hypothesis on  $E_N$ . This implies that  $Y_{s,t} \in E_N \cap \phi(\mathbf{u}_{\mathbf{d}}) = \{0\}$ . By Lemma 2.4, this gives  $U_{s,t} = 0$ . The claim is proved.

In particular, this implies that the action of the group  $U_{\mathbf{d}}$  on  $Y_N$  is free. Thus,  $|E_N| = \frac{|Y_N|}{|U_{\mathbf{d}}|} = |Y_N/U_{\mathbf{d}}|$ . This equality, together with the claim just proved, gives (ii).  $\square$

The following corollary can be seen as a generalization of Riedmann's formula, [8].

**Corollary 2.6.** *Let  $E_N$  be as in the previous proposition. Then,*

$$|\mathcal{F}_{N_{\nu}, \dots, N_1}^M| = \frac{|Aut(M)| \cdot |E_N \cap \mathcal{O}_M|}{\prod_s |Aut(N_s)|},$$

where  $\mathcal{F}_{N_{\nu}, \dots, N_1}^M$  denotes the set of filtrations  $0 = M^{\nu} \subset \dots \subset M^1 \subset M^0 = M$  of the representation  $M$  with successive subquotients  $M^{s-1}/M^s$  isomorphic to  $N_s$  for  $1 \leq s \leq \nu$ .

*Proof.* By construction, we have  $\mathcal{F}_{N_{\nu}, \dots, N_1}^M = p_1^{-1}(M) \cap X_{\overline{N}}$ . Moreover, as  $U_{\mathbf{d}} \subset G_d$ , the previous proposition implies that  $E_N \cap \mathcal{O}_M$  is a transversal slice for the action of  $U_{\mathbf{d}}$  on  $Y_N \cap \mathcal{O}_M$ . The corollary then follows from Proposition 2.3.  $\square$

It is known [10] that there exists a polynomial  $F_{N_{\nu}, \dots, N_1}^M(t) \in \mathbb{Z}[t]$ , called the generalized Hall polynomial, whose value at any prime power  $q$  equals the number of  $\mathbb{F}_q$ -rational points of the variety  $\mathcal{F}_{N_{\nu}, \dots, N_1}^M$ .

### 3. HALL ALGEBRAS AND COEFFICIENTS OF THE BAR AUTOMORPHISM.

3.1. We define the Euler form  $\langle \cdot, \cdot \rangle$  on  $\mathbb{N}I$  by  $\langle d, e \rangle := \sum_{i \in I} d_i e_i - \sum_{\alpha: i \rightarrow j} d_i e_j$ . We suppose in this section that  $k$  is a field with  $q = v^2$  elements for some  $v \in \mathbb{C}$ . The dimension type  $d$  and the representations  $M, N$  are no longer fixed. For all finite sets  $X$  on which  $G_d$  acts, we denote by  $\mathbb{C}_{G_d}[X]$  the set of  $G_d$ -invariant functions from  $X$  to  $\mathbb{C}$ . Define

$$\mathcal{H}_v(Q) = \bigoplus_{d \in \mathbb{N}I} \mathbb{C}_{G_d}[R_d].$$

The space  $\mathcal{H}_v(Q)$  is endowed with a structure of  $\mathbb{N}I$ -graded  $\mathbb{C}$ -algebra by the convolution product :

$$(f \cdot g)(X) = v^{< d, e >} \sum_{U \subset X} f(X/U)g(U), \quad f \in \mathbb{C}_{G_d}[R_d], \quad g \in \mathbb{C}_{G_e}[R_e], \quad X \in R_{d+e},$$

where  $U$  runs over all subrepresentations of  $X$  of dimension type  $e$ . It is known [10] that this product defines the structure of an associative algebra on  $\mathcal{H}_v(Q)$ , which is called the (twisted) Hall algebra of the quiver  $Q$ .

For any representation  $M$  of  $Q$  with isoclass  $\overline{M}$ , let  $e_M = e_{\overline{M}}$  be  $v^{\dim \text{End}N - \dim N}$  times the characteristic function of the orbit  $\mathcal{O}_M$ . It is clear that  $\{e_{\overline{M}}\}_{\overline{M} \in \overline{\text{mod } kQ}}$  is a basis of  $\mathcal{H}_v(Q)$ .

Let  $S_i$  be the simple representation corresponding to the vertex  $i$  in  $I$ . It is known [11] that there exists an isomorphism  $\eta$  from the Hall algebra  $\mathcal{H}_v(Q)$  to the positive part  $U_q(\mathfrak{g})^+$  of the quantum enveloping algebra associated to  $Q$ , such that  $\eta$  maps  $e_{S_i}$  to the canonical generator  $e_i$  of  $U_q(\mathfrak{g})^+$ . Note that the basis  $e_M = e_{\overline{M}}$  is sent to the so-called Poincaré-Birkhoff-Witt basis of  $U_q(\mathfrak{g})^+$  which corresponds to a reduced decomposition of the longest Weyl group element naturally associated to  $Q$ , see [4, 4.12].

3.2. We consider the inner product on  $\mathcal{H}_v(Q)$ , called Green form, defined by

$$(e_M, e_N) = v^{2\dim \text{End}N} a_M^{-1} \delta_{N,M},$$

where  $\delta$  is the Kronecker symbol and  $a_M := a_M(v^2) = |\text{Aut}(M)|$ . Hence, we obtain the dual PBW type basis by setting  $e_M^* = v^{-2\dim \text{End}N} a_M e_M$ .

The following lemma is an easy consequence of the definition of the convolution product in the Hall algebra and of the properties of the decomposition  $N = \bigoplus_s N_s$ .

**Lemma 3.1.** *Suppose that  $N = \bigoplus_s N_s$  is the decomposition of  $N$  into powers of indecomposables as above. Then,*

- (i)  $e_N = e_{N_1} \dots e_{N_\nu}$ ,
- (ii)  $e_N^* = e_{N_1}^* \dots e_{N_\nu}^*$ ,
- (iii)  $e_{N_\nu} \dots e_{N_1} = \sum_M v^{S - \dim \text{End}M} F_{N_\nu, \dots, N_1}^M(v^2) e_M$ , where  $S = \sum_s \dim \text{End}N_s + \sum_{s > t} \langle \dim N_s, \dim N_t \rangle$ .

3.3. Using the basis elements  $e_M$  introduced above, the multiplication in the Hall algebra reads as follows:

$$e_M \cdot e_N = \sum_{\overline{X}} v^{\dim \text{End}M + \dim \text{End}N + \langle \dim M, \dim N \rangle - \dim \text{End}X} F_{M,N}^X(v^2) \cdot e_X.$$

Since the  $F_{M,N}^X$  are polynomials, we can thus take the above formula as the definition of structure constants for a  $\mathbb{Q}(v)$ -algebra, the generic (twisted) Hall algebra [11], which will still be denoted by  $\mathcal{H}_v(Q)$ .

We define a  $\mathbb{Q}$ -linear involution on  $\mathbb{Q}(v)$  by  $\overline{v} = v^{-1}$ . We define on  $\mathcal{H}_v(Q)$  :

- a  $\mathbb{Q}$ -linear involution by  $\overline{e_i} = e_i$ , called the bar involution,
- the  $\mathbb{Q}(v)$ -linear antiinvolution  $\sigma$  by  $\sigma(e_i) = e_i$ .

Denote by  $\omega_{M,N}$  the  $e_N$ -coefficient of  $\overline{e_M}$  in the PBW-basis. It is clear that  $\omega_{M,N}$  is zero if  $M$  and  $N$  do not have the same dimension type. Following Lusztig [4], we use the normalization  $\Omega_{M,N} = v^{\dim \mathcal{O}_N - \dim \mathcal{O}_M} \omega_{M,N} = v^{\dim \text{End}M - \dim \text{End}N} \omega_{M,N}$ . We say that  $M$  degenerates to  $N$  if  $\mathcal{O}_N$  belongs to the closure of  $\mathcal{O}_M$ . The following is proved in [4].

**Lemma 3.2.** *For any two representations  $M, N$  in  $\text{mod } kQ$ , we have:*

- (i) *if  $\Omega_{M,N} \neq 0$ , then  $M$  degenerates to  $N$ ,*
- (ii)  $\Omega_{M,M} = 1$ ,
- (iii)  $\Omega_{M,N} \in \mathbb{Z}[v^{-2}]$ .

We now want to give a geometric interpretation of the polynomial  $\overline{\Omega_{M,N}}$ . This will be provided by Theorem 3.4. The following proposition precises a result of [3, Proposition 3.1], where the formula was asserted up to a power of  $v$ .

**Proposition 3.3.** *Let  $M, N$  be two representations in  $\text{mod}kQ$ , and let  $N = \bigoplus_s N_s$  be the decomposition into powers of indecomposables as above. Then, the polynomial  $\overline{\Omega_{M,N}} \in \mathbb{Z}[v^2]$  is given by*

$$\overline{\Omega_{M,N}} = F_{N_\nu, \dots, N_1}^M(v^2) \frac{\prod_s a_{N_s}}{a_M}.$$

*Proof.* We first calculate  $\omega_{M,N} = (\overline{e_M}, e_N^*)$  by using the adjoint of the bar automorphism for the Green form. From [6, 1.2.10.], we have :

$$\overline{\omega_{M,N}} = (-v)^{-\dim M} v^{-<\dim N, \dim N>} (e_M, \sigma(\overline{e_N^*})).$$

From Lemma 3.1 (ii), this gives :

$$\overline{\omega_{M,N}} = (-v)^{-\dim M} v^{-<\dim N, \dim N>} (e_M, \sigma(\overline{e_{N_\nu}^*}) \dots \sigma(\overline{e_{N_1}^*})).$$

By [2], the elements  $e_{N_s}^*$  belong to the dual canonical basis. Hence, by [7, Lemma 4.3.],

$$\sigma(e_{N_s}^*) = (-v)^{\sum \dim N_s} v^{<\dim N_s, \dim N_s>} e_{N_s}^*$$

for all  $s = 1, \dots, \nu$ . We deduce that

$$\overline{\omega_{M,N}} = v^{-\sum_{s \neq t} <\dim N_s, \dim N_t>} (e_M, e_{N_\nu}^* \dots e_{N_1}^*).$$

It remains to calculate the  $e_M^*$ -component of the product  $e_{N_\nu}^* \dots e_{N_1}^*$  in the dual PBW-basis. This is obtained from Lemma 3.1 :

$$(e_M, e_{N_\nu}^* \dots e_{N_1}^*) = v^T F_{N_\nu, \dots, N_1}^M a_{N_1} \dots a_{N_\nu} a_M^{-1},$$

where  $T = -\sum_s \dim \text{End} N_s + \dim \text{End} M + \sum_{s>t} <\dim N_s, \dim N_t>$ . Now, from the interpretation of  $<\cdot, \cdot>$  as the homological Euler form in  $\text{mod}kQ$ , namely  $<\dim M, \dim N> = \dim \text{Hom}(M, N) - \dim \text{Ext}^1(M, N)$ , we easily obtain

$$\dim \text{End}(N) = \sum_s \dim \text{End}(N_s) + \sum_{s \leq t} <\dim N_s, \dim N_t>,$$

and the claimed formula follows.  $\square$

3.4. Our efforts are rewarded in that we can deduce a geometric interpretation of the coefficient  $\overline{\Omega_{M,N}}$ .

**Theorem 3.4.** *Let  $k$  be the finite field  $\mathbb{F}_{v^2}$ . Fix a dimension type  $d$  in  $\text{NI}$ , and fix representations  $N, M$  in  $R_d$ . Let  $E_N$  be a graded complementary of the tangent space of  $\mathcal{O}_N$  at  $N$  as in Proposition 2.5. Then, the value of the polynomial  $\overline{\Omega_{M,N}}$  at  $v^2$  equals the cardinality of the set  $E_N \cap \mathcal{O}_M$ .*

*Proof.* This is Proposition 3.3 combined with Corollary 2.6.  $\square$

Note that the theorem implies the following curious identity :

**Corollary 3.5.** *Let  $d$  be in  $\text{NI}$  and  $N$  in  $R_d$ . Then,*

$$\sum_P \overline{\Omega_{P,N}} = |E_N| = q^{\dim \text{Ext}^1(N, N)},$$

where  $P$  runs over the set of isoclasses of representations of dimension type  $d$ .

We finish the section with the following remark. The (generalized) Hall polynomials are known to have leading coefficient equal to one. Hence, by the Lang-Weil theorem, all the varieties  $E_N \cap \mathcal{O}_M$  have a unique irreducible component of maximal dimension.

#### 4. THE PREPROJECTIVE VARIETY AND COEFFICIENTS OF THE BAR AUTOMORPHISM.

4.1. For any arrow  $\alpha : i \rightarrow j$  in the quiver  $Q$ , we define  $i(\alpha) = i$  and  $h(\alpha) = j$ . Let  $Q^{op}$  be the opposite quiver, having the same vertices as  $Q$ , and an arrow  $\alpha^* : j \rightarrow i$  for each arrow  $\alpha : i \rightarrow j$  in  $Q$ .

Fix a dimension type  $d = \sum_i d_i i$  in  $NI$ . As above,  $G_d$  acts on  $R_d(Q) = R_d$  and on  $R_d(Q^{op})$ . Note that the map sending a linear map to its adjoint induces an isomorphism  $R_d(Q) \rightarrow R_d(Q^{op})$ ,  $M \mapsto M^*$ .

Let  $\Pi_d$  be the preprojective variety (see [5]):

$$\begin{aligned} \Pi_d := \{ ((M_\alpha)_\alpha, (N_{\alpha^*})_\alpha) \in R_d(Q) \times R_d(Q^{op}), \text{ for all } i \in I : \\ \sum_{\alpha \in Q_1, h(\alpha)=i} N_{\alpha^*} M_\alpha = \sum_{\alpha \in Q_1, i(\alpha)=i} M_\alpha N_{\alpha^*} \} \subset R_d(Q) \times R_d(Q^{op}). \end{aligned}$$

We denote by  $p$  (resp.  $p^{op}$ ) the canonical projection  $\Pi_d \rightarrow R_d(Q)$  (resp.  $\Pi_d \rightarrow R_d(Q^{op})$ ). We also define a twisted projection  $\tilde{p} : \Pi_d \rightarrow R_d(Q)$  by mapping  $(M, N)$  to  $M + N^*$  using the identification  $R_d(Q) \simeq R_d(Q^{op})$  above.

4.2. Observe that, for all  $N$  in  $R_d$ , the identification  $R_d(Q) \rightarrow R_d(Q^{op})$  maps the orbit  $G_d.N$  to  $G_d.N^*$  and the tangent space  $T_N(\mathcal{O}_N)$  to  $T_{N^*}(\mathcal{O}_{N^*})$ .

We consider the non degenerate pairing on  $R_d(Q) \times R_d(Q^{op})$  given by

$$\langle (M_\alpha), (N_{\alpha^*}) \rangle = \sum_{\alpha \in Q_1} \text{Tr}(M_\alpha N_{\alpha^*}).$$

We have the following

**Lemma 4.1.** *With respect to the pairing above, we have  $T_N(\mathcal{O}_N)^\perp = p^{op}(p^{-1}(N))$ , for all  $N$  in  $R_d(Q)$ .*

*Proof.* By Lemma 2.4 (ii), an element  $(f_{\alpha^*})$  of  $R_d(Q^{op})$  belongs to  $T_N(\mathcal{O}_N)^\perp$  if and only if for all  $X$  in  $\mathfrak{g}_d$ , we have

$$\sum_{\alpha \in Q_1} \text{Tr}((X_{h(\alpha)} N_\alpha - N_\alpha X_{i(\alpha)}) f_{\alpha^*}) = 0.$$

By well known properties of the trace form, this is equivalent to

$$\sum_{\alpha \in Q_1} \text{Tr}(X_{h(\alpha)} N_\alpha f_{\alpha^*}) - \text{Tr}(X_{i(\alpha)} f_{\alpha^*} N_\alpha) = 0,$$

thus

$$\sum_{i \in I} X_i \left( \sum_{i(\alpha)=i} N_\alpha f_{\alpha^*} - \sum_{h(\alpha)=i} f_{\alpha^*} N_\alpha \right) = 0.$$

Since the trace form is non-degenerate, this gives

$$\sum_{i(\alpha)=i} N_\alpha f_{\alpha^*} - \sum_{h(\alpha)=i} f_{\alpha^*} N_\alpha = 0,$$

which proves  $(f_{\alpha^*}) \in p^{op}(p^{-1}(N))$  as required.  $\square$

For any morphism  $\pi : X \rightarrow Y$  of  $k$ -varieties, and any function  $f$  (resp.  $g$ ) on  $X$  (resp.  $Y$ ), we define as usual :

$$\begin{aligned}\pi^*(g) : X \rightarrow k, x \mapsto g(\pi(x)), \\ \pi_*(f) : Y \rightarrow k, y \mapsto \sum_{\pi(x)=y} f(x).\end{aligned}$$

For a subset  $A$  of  $R_d$  or  $R_d(Q^{op})$ , we denote by  $1_A$  the corresponding characteristic function. The previous lemma gives the following interpretation of the polynomials  $\overline{\Omega_{M,N}}$  in terms of the geometry of the preprojective variety:

**Proposition 4.2.** *For  $M$  in  $R_d(Q)$  and  $N^*$  in  $R_d(Q^{op})$ , we have*

$$(p^{op})_*(\tilde{p})^*(1_{\mathcal{O}_M})(N^*) = \overline{\Omega_{M,N}}.$$

*Proof.* Obviously,  $(p^{op})_*(\tilde{p})^*(1_{\mathcal{O}_M})$  belongs to  $\mathbb{C}_{G_d}[R_d(Q^{op})]$ .

Fix  $N$  in  $R_d$ . By the definitions,  $(p^{op})_*(\tilde{p})^*(1_{\mathcal{O}_M})(N^*)$  can be rewritten as  $(p^{op})_*(f_M)(N^*)$ , where the function  $f$  on  $\Pi_d$  is defined by  $f_M(A, B^*) = 1$  if  $A + B \in \mathcal{O}_M$ , and 0 otherwise. Hence, by the previous lemma,

$$(p^{op})_*(\tilde{p})^*(1_{\mathcal{O}_M})(N^*) = \sum_{A \in T_{N^*}(\mathcal{O}_{N^*})^\perp} f_M(A, N^*) = |\mathcal{O}_M \cap (N + T_{N^*}(\mathcal{O}_{N^*})^\perp)|.$$

By Theorem 3.4, this gives  $(p^{op})_*(\tilde{p})^*(1_{\mathcal{O}_M})(N^*) = \overline{\Omega_{M,N}}$ .  $\square$

## REFERENCES

- [1] K. Bongartz. On degenerations and extensions of finite dimensional modules. *Adv. Math* 121, 245–287, 1996.
- [2] P. Caldero. A multiplicative property of quantum flag minors. *Representation Theory* 7, 164–176, 2003.
- [3] P. Caldero, R. Schiffler. Rational smoothness of varieties of representations for quivers of Dynkin type. *ArXiv/math.RT/0305149*.
- [4] G. Lusztig. Canonical bases arising from quantized enveloping algebras. *J.Amer.Math.Soc.* 3, 447–498, 1990.
- [5] G. Lusztig. Canonical bases arising from quantized enveloping algebras II, in "Common trends in mathematics and quantum field theories", *Progr. of Theor. Phys. Suppl.* 102, 175–201, 1990, ed. T. Eguchi et. al.
- [6] G. Lusztig. Introduction to quantum groups. *Progress in Mathematics* 110, Birkhäuser, Boston, 1993.
- [7] M. Reineke. Multiplicative properties of dual canonical bases of quantum groups. *J. Algebra* 211, 134–149, 1999.
- [8] C. Riedmann. Lie algebras generated by indecomposables. *Journal of algebra* 170, 526–546, 1994.
- [9] C. M. Ringel. Tame algebras and integral quadratic forms. *Lecture Notes in Mathematics* 1099, Springer, Berlin, 1984.
- [10] C. M. Ringel. Hall algebras. *Topics in Algebra, Part I* (Warsaw, 1988), 433-447. Banach Center Publ. 26, Part 1, Warsaw, 1990.
- [11] C. M. Ringel. Hall algebras and quantum groups. *Invent. Math.* 101, 583–591, 1990.

DÉPARTEMENT DE MATHÉMATIQUES, UNIVERSITÉ CLAUDE BERNARD LYON I, 69622 VILLEURBANNE CEDEX, FRANCE

*E-mail address:* caldero@igd.univ-lyon1.fr

MATHEMATISCHES INSTITUT, UNIVERSITÄT MÜNSTER, 48149 MÜNSTER, GERMANY

*E-mail address:* reinekem@math.uni-muenster.de